Suggested Solution to Assignment 3

1. (a) Write down the tableau and perform pivot operations successively. The pivoting entries are marked with asterisks.

| | $y_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | 3 | 2 | 2 | 15 |
| $x_2$ | 0 | $4^*$ | 5 | 24 |
| $-1$ | 3 | 5 | 4 | $-12$ |

$\longrightarrow$

| | $y_1$ | $x_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $3^*$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 3 |
| $y_2$ | 0 | $\frac{1}{4}$ | $\frac{5}{4}$ | 6 |
| $-1$ | 3 | $-\frac{5}{4}$ | $-\frac{9}{4}$ | $-42$ |

$\longrightarrow$

| | $x_1$ | $x_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{1}{3}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | 1 |
| $y_2$ | 0 | $\frac{1}{4}$ | $\frac{5}{4}$ | 6 |
| $-1$ | $-1$ | $-\frac{3}{4}$ | $-\frac{7}{4}$ | $-45$ |

The independent variables are $y_3, y_4, y_5$ and the basic variables are $y_1, y_2$. The basic solution is

$$y_3 = y_4 = y_5 = 0, \quad y_1 = 1, \quad y_2 = 6.$$

Thus an optimal vector for the primal problem is

$$(y_1, y_2, y_3) = (1, 6, 0)$$

The maximum value of $f$ is 45.

The dual problem is

$$\min g = 15x_1 + 24x_2 + 12$$

subject to
$$3x_1 \geq 3$$
$$2x_1 + 4x_2 \geq 5$$
$$2x_1 + 5x_2 \geq 4$$

From the above tableau, the independent variables are $x_3, x_4$ and the basic variables are $x_1, x_2, x_5$. The basic solution is

$$x_3 = x_4 = 0, \quad x_1 = 1, \quad x_2 = \frac{3}{4}, \quad x_5 = \frac{7}{4}.$$

Therefore an optimal vector for the dual problem is

$$(x_1, x_2) = (1, \frac{3}{4})$$

The minimum value of $g$ is 45 which is equal to the maximum value of $f$.

(b) Write down the tableau and perform pivot operations successively.

The pivoting entries are marked with asterisks.

| | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $-1$ |
|---|---|---|---|---|---|
| $x_1$ | 3 | 1 | 1 | 4 | 12 |
| $x_2$ | 1 | $-3$ | 2 | 3 | 7 |
| $x_3$ | 2 | $1^*$ | 3 | $-1$ | 10 |
| $-1$ | 2 | 4 | 3 | 1 | 0 |

$\longrightarrow$

| | $y_1$ | $x_3$ | $y_3$ | $y_4$ | $-1$ |
|---|---|---|---|---|---|
| $x_1$ | 1 | $-1$ | $-2$ | $5^*$ | 2 |
| $x_2$ | 7 | 3 | 11 | 0 | 37 |
| $y_2$ | 2 | 1 | 3 | $-1$ | 10 |
| $-1$ | $-6$ | $-4$ | $-9$ | 5 | $-40$ |

$\longrightarrow$

| | $y_1$ | $x_3$ | $y_3$ | $x_1$ | $-1$ |
|---|---|---|---|---|---|
| $y_4$ | $\frac{1}{5}$ | $-\frac{1}{5}$ | $-\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ |
| $x_2$ | 7 | 3 | 11 | 0 | 37 |
| $y_2$ | $\frac{11}{5}$ | $\frac{4}{5}$ | $\frac{13}{5}$ | $\frac{1}{5}$ | $\frac{52}{5}$ |
| $-1$ | $-7$ | $-3$ | $-7$ | $-1$ | $-42$ |

The independent variables are $y_1, y_3, y_5, y_7$ and the basic variables are $y_2, y_4, y_6$. The basic solution is

$$y_1 = y_3 = y_5 = y_7 = 0, \quad y_2 = \frac{52}{5}, \quad y_4 = \frac{2}{5}, \quad y_6 = 37$$

Thus an optimal vector for the primal problem is

$$(y_1, y_2, y_3, y_4) = \left(0, \frac{52}{5}, 0, \frac{2}{5}\right)$$

The maximum value of $f$ is 42.

The dual problem is

$$\min \ g = 12x_1 + 7x_2 + 10x_3$$

$$\text{subject to} \quad \begin{aligned} 3x_1 + x_2 + 2x_3 &\geq 2 \\ x_1 - 3x_2 + x_3 &\geq 4 \\ x_1 + 2x_2 + 3x_3 &\geq 3 \\ 4x_1 + 3x_2 - x_3 &\geq 1 \end{aligned}$$

From the above tableau, the independent variables are $x_2, x_5, x_7$ and the basic variables are $x_1, x_3, x_4, x_6$. The basic solution is

$$x_2 = x_5 = x_7 = 0, \quad x_1 = 1, \quad x_3 = 3, \quad x_4 = x_6 = 7$$

Therefore an optimal vector for the dual problem is

$$(x_1, x_2, x_3) = (1, 0, 3).$$

The minimum value of $g$ is 42 which is equal to the maximum value of $f$.

2.(a) Add $k=3$ to every entry to get

$$\begin{pmatrix} 5 & 0 & 6 \\ 1 & 6 & 4 \\ 4 & 4 & 8 \end{pmatrix}$$

Applying simplex algorithm, we have

| | $y_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $5^*$ | $0$ | $6$ | $1$ |
| $x_2$ | $1$ | $6$ | $4$ | $1$ |
| $x_3$ | $4$ | $4$ | $8$ | $1$ |
| $-1$ | $1$ | $1$ | $1$ | $0$ |

$\longrightarrow$

| | $x_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{1}{5}$ | $0$ | $\frac{6}{5}$ | $\frac{1}{5}$ |
| $x_2$ | $-\frac{1}{5}$ | $6$ | $\frac{14}{5}$ | $\frac{4}{5}$ |
| $x_3$ | $-\frac{4}{5}$ | $4^*$ | $\frac{16}{5}$ | $\frac{1}{5}$ |
| $-1$ | $-\frac{1}{5}$ | $1$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ |

$\longrightarrow$

| | $x_1$ | $x_3$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{1}{5}$ | $0$ | $\frac{6}{5}$ | $\frac{1}{5}$ |
| $x_2$ | $1$ | $-\frac{3}{2}$ | $-2$ | $\frac{1}{2}$ |
| $y_2$ | $-\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{4}{5}$ | $\frac{1}{20}$ |
| $-1$ | $0$ | $-\frac{1}{4}$ | $-1$ | $-\frac{1}{4}$ |

The independent variables are $x_2, x_4, x_5$; $y_3, y_4, y_6$ and the basic variables are $x_1, x_3, x_6$; $y_1, y_2, y_5$. The basic solution is

$$x_2 = x_4 = x_5 = 0, \ x_1 = 0, \ x_3 = \frac{1}{4}, \ x_6 = 1$$

$$y_3 = y_4 = y_6 = 0, \ y_1 = \frac{1}{5}, \ y_2 = \frac{1}{20}, \ y_5 = \frac{1}{2}$$

The optimal value is $d = \frac{1}{4}$.

Therefore a maximin strategy for the row player is

$$\vec{p} = \frac{1}{d}(x_1, x_2, x_3) = (0, 0, 1)$$

a minimax strategy for the column player is

$$\vec{q} = \frac{1}{d}(y_1, y_2, y_3) = \left(\frac{4}{5}, \frac{1}{5}, 0\right)$$

The value of the game is $v = \frac{1}{d} - k = 1$.

(b) Add $k=5$ to every entry to get

$$\begin{pmatrix} 8 & 6 & 0 \\ 4 & 3 & 11 \\ 3 & 4 & 7 \end{pmatrix}$$

Applying simplex algorithm, we have

| | $y_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $8^*$ | $6$ | $0$ | $1$ |
| $x_2$ | $4$ | $3$ | $11$ | $1$ |
| $x_3$ | $3$ | $4$ | $7$ | $1$ |
| $-1$ | $1$ | $1$ | $1$ | $0$ |

$\longrightarrow$

| | $x_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{1}{8}$ | $\frac{3}{4}$ | $0$ | $\frac{1}{8}$ |
| $x_2$ | $-\frac{1}{2}$ | $0$ | $11^*$ | $\frac{1}{2}$ |
| $x_3$ | $-\frac{3}{8}$ | $\frac{7}{4}$ | $7$ | $\frac{5}{8}$ |
| $-1$ | $-\frac{1}{8}$ | $\frac{1}{4}$ | $1$ | $-\frac{1}{8}$ |

$\longrightarrow$

| | $x_1$ | $y_2$ | $x_2$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{1}{8}$ | $\frac{3}{4}^*$ | $0$ | $\frac{1}{8}$ |
| $y_3$ | $-\frac{1}{22}$ | $0$ | $\frac{1}{11}$ | $\frac{1}{22}$ |
| $x_3$ | $-\frac{5}{88}$ | $\frac{7}{4}$ | $-\frac{7}{11}$ | $\frac{27}{88}$ |
| $-1$ | $-\frac{7}{88}$ | $\frac{1}{4}$ | $-\frac{1}{11}$ | $-\frac{15}{88}$ |

$$\longrightarrow \quad \begin{array}{c|ccc|c} & x_1 & y_1 & x_2 & -1 \\ \hline y_2 & \frac{1}{6} & \frac{4}{3} & 0 & \frac{1}{6} \\ y_3 & -\frac{1}{22} & 0 & \frac{1}{11} & \frac{1}{22} \\ x_3 & -\frac{23}{66} & -\frac{7}{3} & -\frac{7}{11} & \frac{35}{132} \\ \hline -1 & -\frac{4}{33} & -\frac{1}{3} & -\frac{1}{11} & -\frac{7}{33} \end{array}$$

Thus $x_3 = x_5 = x_6 = 0$, $x_1 = \frac{4}{33}$, $x_2 = \frac{1}{11}$, $x_4 = \frac{1}{3}$

$y_1 = y_4 = y_5 = 0$, $y_2 = \frac{1}{6}$, $y_3 = \frac{1}{22}$, $y_6 = \frac{35}{132}$.

$d = \frac{7}{33}$.

Therefore a maximin strategy for the row player is

$$\vec{p} = \frac{1}{d}(x_1, x_2, x_3) = \left(\frac{4}{7}, \frac{3}{7}, 0\right)$$

a minimax strategy for the column player is

$$\vec{q} = \frac{1}{d}(y_1, y_2, y_3) = \left(0, \frac{11}{14}, \frac{3}{14}\right)$$

The value of the game is $v = \frac{1}{d} - k = -\frac{2}{7}$.

(c) Add $k = 2$ to every entry to get

$$\begin{pmatrix} 5 & 2 & 3 \\ 1 & 4 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

Applying simplex algorithm, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & -1 \\ \hline x_1 & 5^* & 2 & 3 & 1 \\ x_2 & 1 & 4 & 0 & 1 \\ x_3 & 2 & 3 & 1 & 1 \\ \hline -1 & 1 & 1 & 1 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & y_2 & y_3 & -1 \\ \hline y_1 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{1}{5} \\ x_2 & -\frac{1}{5} & \frac{18}{5}^* & -\frac{3}{5} & \frac{4}{5} \\ x_3 & -\frac{2}{5} & \frac{11}{5} & -\frac{1}{5} & \frac{3}{5} \\ \hline -1 & -\frac{1}{5} & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & y_3 & -1 \\ \hline y_1 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{3}^* & \frac{1}{9} \\ y_2 & -\frac{1}{18} & \frac{5}{18} & -\frac{1}{6} & \frac{2}{9} \\ x_3 & -\frac{5}{18} & -\frac{11}{18} & \frac{1}{6} & \frac{1}{9} \\ \hline -1 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{3} \end{array}$$

$$\longrightarrow \quad \begin{array}{c|ccc|c} & x_1 & x_2 & y_1 & -1 \\ \hline y_3 & \frac{1}{3} & -\frac{1}{6} & \frac{3}{2} & \frac{1}{6} \\ y_2 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ x_3 & -\frac{1}{3} & -\frac{7}{12} & -\frac{1}{4} & \frac{1}{12} \\ \hline -1 & -\frac{1}{3} & -\frac{1}{12} & -\frac{3}{4} & -\frac{5}{12} \end{array}$$

Thus $x_3 = x_5 = x_6 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{12}$, $x_4 = \frac{3}{4}$

$y_1 = y_4 = y_5 = 0$, $y_2 = \frac{1}{4}$, $y_3 = \frac{1}{6}$, $y_6 = \frac{1}{12}$

$d = \frac{5}{12}$

Therefore a maximin strategy for the row player is

$$\vec{p} = \frac{1}{d}(x_1, x_2, x_3) = \left(\frac{4}{5}, \frac{1}{5}, 0\right)$$

A minimax strategy for the column player is $\vec{q} = \frac{1}{d}(y_1, y_2, y_3) = \left(0, \frac{3}{5}, \frac{2}{5}\right)$.

The value of the game is $v = \frac{1}{d} - k = \frac{2}{5}$.

(d) Add $k=3$ to every entry to get $\begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 6 \\ 1 & 5 & 3 \end{pmatrix}$

Applying simplex algorithm, we have

| | $y_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $5^*$ | $3$ | $1$ | $1$ |
| $x_2$ | $2$ | $0$ | $6$ | $1$ |
| $x_3$ | $1$ | $5$ | $3$ | $1$ |
| $-1$ | $1$ | $1$ | $1$ | $0$ |

$\longrightarrow$

| | $x_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |
| $x_2$ | $-\frac{2}{5}$ | $-\frac{6}{5}$ | $\frac{28}{5}^*$ | $\frac{3}{5}$ |
| $x_3$ | $-\frac{1}{5}$ | $\frac{22}{5}$ | $\frac{14}{5}$ | $\frac{4}{5}$ |
| $-1$ | $-\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $-\frac{1}{5}$ |

$\longrightarrow$

| | $x_1$ | $y_2$ | $x_2$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{3}{14}$ | $\frac{9}{14}$ | $-\frac{1}{28}$ | $\frac{5}{28}$ |
| $y_3$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{5}{28}$ | $\frac{3}{28}$ |
| $x_3$ | $0$ | $5^*$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| $-1$ | $-\frac{1}{7}$ | $\frac{4}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ |

$\longrightarrow$

| | $x_1$ | $x_3$ | $x_2$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{3}{14}$ | $-\frac{9}{70}$ | $\frac{1}{35}$ | $\frac{4}{35}$ |
| $y_3$ | $-\frac{1}{14}$ | $-\frac{3}{70}$ | $\frac{11}{70}$ | $\frac{9}{70}$ |
| $y_2$ | $0$ | $\frac{1}{5}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ |
| $-1$ | $-\frac{1}{7}$ | $-\frac{4}{35}$ | $-\frac{3}{35}$ | $-\frac{12}{35}$ |

Thus $x_4=x_5=x_6=0$, $x_1=\frac{1}{7}$, $x_2=\frac{3}{35}$, $x_3=\frac{4}{35}$

$y_4=y_5=y_6=0$, $y_1=\frac{4}{35}$, $y_2=\frac{1}{10}$, $y_3=\frac{9}{70}$

$d=\frac{12}{35}$.

Therefore a maximum strategy is $\vec{p}=\frac{1}{d}(x_1,x_2,x_3)=(\frac{5}{12},\frac{1}{4},\frac{1}{3})$.

A minimax strategy for column player is $\vec{q}=\frac{1}{d}(y_1,y_2,y_3)=(\frac{1}{3},\frac{7}{24},\frac{3}{8})$

The value of the game is $v=\frac{1}{d}-k=-\frac{1}{12}$.

(e) Add $k=2$ to every entry to get $\begin{pmatrix} 3 & 1 & 3 \\ 0 & 2 & 1 \\ 3 & 0 & 4 \\ 1 & 3 & 0 \end{pmatrix}$

Applying simplex algorithm, we have

| | $y_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $3^*$ | $1$ | $3$ | $1$ |
| $x_2$ | $0$ | $2$ | $1$ | $1$ |
| $x_3$ | $3$ | $0$ | $4$ | $1$ |
| $x_4$ | $1$ | $3$ | $0$ | $1$ |
| $-1$ | $1$ | $1$ | $1$ | $0$ |

$\longrightarrow$

| | $x_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $1$ | $\frac{1}{3}$ |
| $x_2$ | $0$ | $2$ | $1$ | $1$ |
| $x_3$ | $-1$ | $-1$ | $1$ | $0$ |
| $x_4$ | $-\frac{1}{3}$ | $\frac{8}{3}^*$ | $-1$ | $\frac{2}{3}$ |
| $-1$ | $-\frac{1}{3}$ | $\frac{2}{3}$ | $0$ | $-\frac{1}{3}$ |

$\longrightarrow$

| | $x_1$ | $x_4$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $y_1$ | $\frac{9}{24}$ | $-\frac{1}{8}$ | $\frac{9}{8}^*$ | $\frac{1}{4}$ |
| $x_2$ | $\frac{1}{4}$ | $-\frac{3}{4}$ | $\frac{7}{4}$ | $\frac{1}{2}$ |
| $x_3$ | $-\frac{9}{8}$ | $\frac{3}{8}$ | $\frac{5}{8}$ | $\frac{1}{4}$ |
| $y_2$ | $-\frac{1}{8}$ | $\frac{3}{8}$ | $-\frac{3}{8}$ | $\frac{1}{4}$ |
| $-1$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{1}{4}$ | $-\frac{1}{2}$ |

$\longrightarrow$

| | $x_1$ | $x_4$ | $y_1$ | $-1$ |
|---|---|---|---|---|
| $y_3$ | $\frac{1}{3}$ | $-\frac{1}{9}$ | $\frac{8}{9}$ | $\frac{2}{9}$ |
| $x_2$ | $-\frac{1}{3}$ | $-\frac{5}{9}$ | $-\frac{14}{9}$ | $\frac{1}{9}$ |
| $x_3$ | $-\frac{4}{3}$ | $\frac{4}{9}$ | $-\frac{5}{9}$ | $\frac{1}{9}$ |
| $y_2$ | $0$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $-1$ | $-\frac{1}{3}$ | $-\frac{2}{9}$ | $-\frac{2}{9}$ | $-\frac{5}{9}$ |

Thus $x_2=x_3=x_6=x_7=0$, $x_1=\frac{1}{3}$, $x_4=\frac{2}{9}$, $x_5=\frac{2}{9}$

$y_1=y_4=y_7=0$, $y_2=\frac{1}{3}$, $y_3=\frac{2}{9}$, $y_5=\frac{1}{9}$, $y_6=\frac{1}{9}$

$d=\frac{5}{9}$.

Therefore a maximum strategy is $\vec{p}=\frac{1}{d}(x_1,x_2,x_3,x_4)=(\frac{3}{5},0,0,\frac{2}{5})$.

A minimax strategy is $\vec{q}=\frac{1}{d}(y_1,y_2,y_3)=(0,\frac{3}{5},\frac{2}{5})$. The value is $v=\frac{1}{d}-k=-\frac{1}{5}$.

(f) Add $k=3$ to every entry to get

$$\begin{pmatrix} 0 & 5 & 3 \\ 4 & 1 & 2 \\ 2 & 3 & 5 \\ 4 & 4 & 0 \end{pmatrix}$$

Applying simplex algorithm, we have

| | $y_1$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $0$ | $5$ | $3$ | $1$ |
| $x_2$ | $4^*$ | $1$ | $2$ | $1$ |
| $x_3$ | $2$ | $3$ | $5$ | $1$ |
| $x_4$ | $4$ | $4$ | $0$ | $1$ |
| $-1$ | $1$ | $1$ | $1$ | $0$ |

$\longrightarrow$

| | $x_2$ | $y_2$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $0$ | $5$ | $3$ | $1$ |
| $y_1$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
| $x_3$ | $-\frac{1}{2}$ | $\frac{5}{2}$ | $4$ | $\frac{1}{2}$ |
| $x_4$ | $-1$ | $3^*$ | $-2$ | $0$ |
| $-1$ | $-\frac{1}{4}$ | $-\frac{3}{4}$ | $\frac{1}{2}$ | $-\frac{1}{4}$ |

$\longrightarrow$

| | $x_2$ | $x_4$ | $y_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $\frac{5}{3}$ | $-\frac{5}{3}$ | $\frac{19}{3}$ | $1$ |
| $y_1$ | $\frac{1}{3}$ | $-\frac{1}{12}$ | $\frac{2}{3}$ | $\frac{1}{4}$ |
| $x_3$ | $\frac{1}{3}$ | $-\frac{5}{6}$ | $\frac{17}{3}^*$ | $\frac{1}{2}$ |
| $y_2$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | $-\frac{2}{3}$ | $0$ |
| $-1$ | $0$ | $-\frac{1}{4}$ | $1$ | $-\frac{1}{4}$ |

$\longrightarrow$

| | $x_2$ | $x_4$ | $x_3$ | $-1$ |
|---|---|---|---|---|
| $x_1$ | $\frac{22}{17}$ | $-\frac{25}{34}$ | $-\frac{19}{17}$ | $\frac{15}{34}$ |
| $y_1$ | $\frac{5}{17}$ | $\frac{1}{68}$ | $-\frac{2}{17}$ | $\frac{13}{68}$ |
| $y_3$ | $\frac{1}{17}$ | $-\frac{5}{34}$ | $\frac{3}{17}$ | $\frac{3}{34}$ |
| $y_2$ | $-\frac{5}{17}$ | $\frac{4}{17}$ | $\frac{2}{17}$ | $\frac{1}{17}$ |
| $-1$ | $-\frac{1}{17}$ | $-\frac{7}{68}$ | $-\frac{3}{17}$ | $-\frac{23}{68}$ |

Thus $x_1=x_5=x_6=x_7=0$, $x_2=\frac{1}{17}$, $x_4=\frac{7}{68}$, $x_3=\frac{3}{17}$.

$y_5=y_6=y_7=0$, $y_1=\frac{13}{68}$, $y_2=\frac{1}{17}$, $y_3=\frac{3}{34}$, $y_4=\frac{15}{34}$

$d=\frac{23}{68}$.

Therefore a maximin strategy for the row player is $\vec{p}=\left(0, \frac{4}{23}, \frac{12}{23}, \frac{7}{23}\right)$.

A minimax strategy for the column player is $\vec{q}=\left(\frac{13}{23}, \frac{4}{23}, \frac{6}{23}\right)$.

The value of the game is $V=\frac{1}{d}-k=-\frac{1}{23}$.

3. Pf (a) For $x_1, x_2 \in C_1 \cap C_2$, $\lambda \in [0,1]$. $y \triangleq \lambda x_1 + (1-\lambda) x_2$

Due to convexity of $C_1, C_2$, $\lambda x_1 + (1-\lambda) x_2 \in C_1$, $\lambda x_1 + (1-\lambda) x_2 \in C_2$.

That is, $y \in C_1 \cap C_2$. Hence $C_1 \cap C_2$ is convex. $\square$

(b) For $x = x_1 + x_2 \in C_1 + C_2$, $y = y_1 + y_2 \in C_1 + C_2$, $\lambda \in [0,1]$, where $x_1, y_1 \in C_1$, $x_2, y_2 \in C_2$.

$z := \lambda x + (1-\lambda) y = \lambda x_1 + (1-\lambda) y_1 + \lambda x_2 + (1-\lambda) y_2$

Since $C_1, C_2$ are convex, we have

$$\lambda x_1 + (1-\lambda) y_1 \in C_1, \quad \lambda x_2 + (1-\lambda) y_2 \in C_2$$

So $z = \lambda x + (1-\lambda) y \in C_1 + C_2$. Hence $C_1 + C_2$ is convex. $\square$

4. Pf. Let the set of maximin strategy for the row player of A be C, and let $v$ be the value of the game with game matrix A.

Suppose $\vec{u}, \vec{v} \in C$, $\vec{u}=(u_1, \cdots u_n)$, $\vec{v}=(v_1, \cdots v_n)$ such that $\vec{u}A\vec{y}^T \geq v$ and $\vec{v}A\vec{y}^T \geq v$ for any $\vec{y} \in P^m$, where $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i = 1$.

For $\lambda \in [0,1]$, let $\vec{z} = \lambda\vec{u} + (1-\lambda)\vec{v}$, then $z_i = \lambda u_i + (1-\lambda)v_i$.

Hence $\sum_{i=1}^n z_i = \lambda \sum_{i=1}^n u_i + (1-\lambda)\sum_{i=1}^n v_i = \lambda + (1-\lambda) = 1$.

$$\begin{aligned}\vec{z}A\vec{y}^T = [\lambda\vec{u} + (1-\lambda)\vec{v}]A\vec{y}^T &= \lambda\vec{u}A\vec{y}^T + (1-\lambda)\vec{v}A\vec{y}^T \\ &\geq v\lambda + (1-\lambda)v \\ &= v. \quad \forall \vec{y} \in P^m.\end{aligned}$$

So $\vec{z} \in C$. That is, C is convex.

5. (a) $\exists \lambda \in [0,1]$ such that $\vec{z} = \lambda\vec{x} + (1-\lambda)\vec{y}$.

Since $\vec{z}$ is orthogonal to $\vec{x}-\vec{y}$, we have

$$\langle \vec{x}-\vec{y}, \vec{z} \rangle = 0, \text{ i.e. } \langle \vec{x}-\vec{y}, \lambda\vec{x} + (1-\lambda)\vec{y} \rangle = 0$$

$$\Rightarrow \lambda\|\vec{x}\|^2 - \lambda\langle \vec{x}, \vec{y} \rangle + (1-\lambda)\langle \vec{x}, \vec{y} \rangle - (1-\lambda)\|\vec{y}\|^2 = 0$$

$$\Rightarrow \lambda(\|\vec{x}\|^2 - 2\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2) = \|\vec{y}\|^2 - \langle \vec{x}, \vec{y} \rangle$$

$$\Rightarrow \lambda = \frac{\|\vec{y}\|^2 - \langle \vec{x}, \vec{y} \rangle}{\|\vec{x}-\vec{y}\|^2}, \quad 1-\lambda = \frac{\langle \vec{x}, \vec{x}-\vec{y} \rangle}{\|\vec{x}-\vec{y}\|^2} = \frac{\|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle}{\|\vec{x}-\vec{y}\|^2}.$$

Hence $$\vec{z} = \frac{\|\vec{y}\|^2 - \langle \vec{x}, \vec{y} \rangle}{\|\vec{x}-\vec{y}\|^2}\vec{x} + \frac{\|\vec{x}\|^2 - \langle \vec{x}, \vec{y} \rangle}{\|\vec{x}-\vec{y}\|^2}\vec{y}.$$

Pf. (b) If $\langle \vec{x}, \vec{y} \rangle < 0$, then

$$\langle \vec{y}-\vec{x}, \vec{y} \rangle = \|\vec{y}\|^2 - \langle \vec{x}, \vec{y} \rangle > 0, \quad \langle \vec{x}-\vec{y}, \vec{x} \rangle = \|\vec{x}\|^2 - \langle \vec{y}, \vec{x} \rangle > 0.$$

Hence $\vec{z}$ lies on the line segment joining $\vec{x}$ and $\vec{y}$.

So $\vec{z} \in C$ because of convexity of C.

6. Sol1: We suppose that $\vec{0} \notin C$.

By hypothesis, $C$ is the convex hull of $\{\vec{a_1}, \vec{a_2}, \cdots, \vec{a_n}, \vec{e_1}, \vec{e_2}, \cdots \vec{e_m}\}$

Then by lemma 3.4.6 of lecture notes, there exists $\vec{z} = (z_1, z_2, \cdots z_m) \in \mathbb{R}^m$

such that $\langle \vec{z}, \vec{y} \rangle > 0$ for any $\vec{y} \in C$.

In particular, we have $\langle \vec{z}, \vec{e_i} \rangle = z_i > 0$ for any $i = 1, 2, \cdots m$.

Then we take $\vec{x} = \dfrac{\vec{z}}{z_1 + z_2 + \cdots + z_m} \in P^m$.

And we have
$$\langle \vec{x}, \vec{a_j} \rangle = \frac{\langle \vec{z}, \vec{a_j} \rangle}{z_1 + z_2 + \cdots + z_m} > 0 \text{ for any } j = 1, 2, \cdots n.$$

which means $\vec{x} A > \vec{0}$.

Let $\alpha > 0$ be the smallest coordinate of the vector $\vec{x} A$ and we have
$$V_r(A) \geq \min_{y \in P^m} \vec{x} A y^T \geq \alpha > 0.$$

It has been proved that $V_r(A) \leq V_c(A)$ for any matrix $A$ (Theorem 3.4.3 of lecture notes).

Hence $V_c(A) \geq V_r(A) > 0$, which contradicts with $V_c(A) \leq 0$.

Therefore, if $V_c(A) \leq 0$, then $\vec{0} \in C$.

Sol 2: By the definition of $V_c(A)$, if $V_c(A) \leq 0$, then $A \vec{y}^T = \begin{pmatrix} -\lambda_{n+1} \\ -\lambda_{n+2} \\ \vdots \\ -\lambda_{n+m} \end{pmatrix} \leq \vec{0}, \forall \vec{y} \in P^n$

that is, $\lambda_{n+1}, \cdots \lambda_{n+m} \geq 0$

Since we can write $A \vec{y}^T - \begin{pmatrix} -\lambda_{n+1} \\ -\lambda_{n+2} \\ \vdots \\ -\lambda_{n+m} \end{pmatrix} = A \vec{y}^T + \begin{pmatrix} \lambda_{n+1} \\ \lambda_{n+2} \\ \vdots \\ \lambda_{n+m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ as

$$y_1 \vec{a_1} + y_2 \vec{a_2} + \cdots + y_n \vec{a_n} + \lambda_{n+1} \vec{e_1} + \lambda_{n+2} \vec{e_2} + \cdots + \lambda_{n+m} \vec{e_m} = \vec{0}, (*)$$

where $\vec{y} = (y_1, y_2, \cdots y_n) \in P^n$

If we take $s = y_1 + y_2 + \cdots + y_n + \lambda_{n+1} + \lambda_{n+2} + \cdots + \lambda_{n+m}$, then divided by $s$ on both sides of $(*)$, we have $\frac{y_1}{s} \vec{a_1} + \cdots + \frac{y_n}{s} \vec{a_n} + \frac{\lambda_{n+1}}{s} \vec{e_1} + \cdots + \frac{\lambda_{n+m}}{s} \vec{e_m} = \vec{0}$ (#), where $\frac{y_1}{s}, \cdots \frac{y_n}{s}, \frac{\lambda_{n+1}}{s}, \cdots \frac{\lambda_{n+m}}{s} \geq 0$ and $\frac{y_1}{s} + \cdots + \frac{y_n}{s} + \frac{\lambda_{n+1}}{s} + \cdots + \frac{\lambda_{n+m}}{s} = 1$.

Since $C = \mathrm{Conv}(\vec{a_1}, \cdots \vec{a_n}, \vec{e_1}, \cdots \vec{e_m})$, we conclude from (#) that $\vec{0} \in C$.